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ON GROUP DIVISIBLE DESIGNS

D. K. Ghosh

Marwadi University, Rajkot-Morbi Road, Gujarat, India

ABSTRACT

In this article we have developed various methods for the construction of Group Divisible (GD) designs using (i) several series of balanced incomplete block designs, (ii) Kroneckker product of designs. Further we have also obtained GD designs using 2^n and 3^n symmetrical factorial experiments. However, all the methods of construction of GD designs are supported by suitable examples.

KEYWORDS: AND PHRASES: *Kroneckker Product, Incidence Matrix, Singular Group Divisible Designs, Semi Regular Group Divisible Designs and Regular Group Divisible Designs*

Article History

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1 INTRODUCTION

For the use of biological and agricultural experiments, Yates (1936) introduced a new design called balanced incomplete block (BIB) design. Methods of construction, properties and analysis of BIB design was later discussed in detail by Bose (1939). BIB design is not available for all required parameters. Another drawback of BIB design is a large number of replications. Moreover, when experimental units are very large especially in agricultural experiments, we generally prefer to incomplete block designs. The easiest and suitable incomplete block design is a balanced incomplete block (BIB) design. However, the main drawback of BIB designs is that they are not always available for the required number of treatments. To overcome such problems Bose and Nair (1939) introduced another class of incomplete block design as it requires smaller number of replications. Which is called partially balanced incomplete block designs with m associate classes, abbreviated as PBIB(m). Bose and Shimamoto (1952) classified PBIB (2) designs into five types and stated Group Divisible (GD) designs to be the simplest of all. Bose and Connor (1952) further classified GD designs into Singular, Semi-Regular and Regular GD designs on the basis of their combinatorial properties. Several authors like John and Turner (1977), John (1977), Kageyama and Tanaka (1981), and Kageyama (1985) have suggested various methods of construction of semi-regular group divisible designs. Moreover, Ghosh and Das (1989), Ghosh and Bhimani (1990) have obtained some more SR GD designs. Ghosh and Das (1993) carried out the construction of group divisible designs with partial balance for group comparisons. Jagdish Prasad et. al. (2011) constructed several semi regular and regular group divisible designs. Recently Sharma et. al. (2016) discussed the characterization of group divisible designs. Very recently Ghosh and Sinojia (2020a, b) developed the construction of semi-regular and singular group divisible designs using Hadmard matrix and varietal codes of factorial experiments. In this article we discuss several other methods for construction of group divisible designs.

2. CONSTRUCTION OF SERIES OF INCOMPLETE BLOCK DESIGNS

3.1 Singular Group Divisible Designs from 3^n Factorial Experiments

Singular Group Divisible designs can be constructed using 3^n factorial experiments in the following way: Develop 3^n combination using n factors each at 3 levels which gives a matrix with n columns and 3^n rows. Delete a row whose all elements are zero so number of rows becomes 3^n - 1. Denote this matrix by N. Develop extra columns by taking all possible combination of n columns with mod 3. Since elements of GF (3) are 0, 1 and 2 and hence, replace 2 by 1 as singular group divisible design is a binary. Consider matrix N as incidence matrix of an incomplete block designs. Assuming columns as blocks and rows as number of treatments, incidence matrix N gives a Singular Group Divisible designs with (3^n – 1) treatments. This is explained with the example shown in Example 3.1.

Example 3.1 Construct a Singular Group Divisible designs with parameters v = 8, b =4, r = 3, k = 6, λ_1 = 3, n_1= 1, λ_2 = 2, n_2 = 6, n = 2, m = 4.

Since, v = 3^n -1 = 8, this implies n = 2. Developing 9 combinations with X_1, X_2, X_1+X_2 and X_1+2X_2 and then using section 3.1, we can construct the required design.

$$
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 2 \\
0 & 2 & 2 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 2 & 0 \\
1 & 2 & 0 & 2 \\
2 & 0 & 2 & 2 \\
2 & 1 & 0 & 1 \\
2 & 2 & 1 & 0
\end{array}
\rightarrow N =
\begin{array}{cccc}
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0
\end{array}
$$

Blocks of the SGD design with parameters v = 8, b =4, r = 3, k = 6, λ_1 = 3, n_1= 1, λ_2 = 2, n_2 = 6, n = 2, m = 4 along with $P_{ij}^1 = \begin{bmatrix} 0 & 0 \\ 0 & 6 \end{bmatrix}$ and $P_{ij}^2 = \begin{bmatrix} 0 & 1 \\ 1 & 4 \end{bmatrix}$ are

(3, 4, 5, 6, 7, 8), (1, 2, 4, 5, 7, 8), (1, 2, 3, 4, 6, 8), (1, 2, 3, 5, 6, 7).

3.2 Partially Balanced Incomplete Block Design using Factorial Designs: Let X_1, X_2, …, X_n be n factors each at two levels. Therefore, the symmetrical factorial experiment is 2^n. Develop 2^n treatment combinations of n factors by combining two levels 0 and 1 and then keep them in a block. Accordingly, it has n columns and 2^n rows. Add (2^n – n -1) more columns with n columns. Consider these extra columns as number of extra factors. Therefore, we obtain n + (2^n – n - 1) = (2^n -1) columns and 2^n rows. Denote these rows and columns by a matrix M. Elements under (2^n - n -1) columns of this matrix M are filled up with all possible linear combinations of n factors row wise. Which satisfies $X_i + X_j$ = 0 mod 2, $X_i + X_j + X_k$ = 0 mod 2, …, $X_i + X_j + X_k + \ldots + X_n$ = 0 mod 2, where i ≠ j ≠ k ≠ … ≠ n = 1, 2, …, n. Delete first row of matrix M whose all elements are zero. Repeat this matrix p times, hence the new matrix has p (2^n – 1) rows and (2^n -1) columns, respectively. Denote this matrix by N. Considering N as incidence matrix of an incomplete block design a series of singular group divisible design is obtained with parameters v = p (2^n – 1), b = 2^n - 1, r = 2^{n-1}, k = p 2^{n-1}, n_1 = 1, λ_1 = 2^{n-1}= r, n_2 = 4(2^{n-1} – 1), λ_2 = 2^{n-2}, n = 2, m = 2^n - 1 = b.

Example 3.1 Construct a singular group divisible design with parameters v = 14, b = 7, r = 4, k = 8, n_1 = 1, λ_1 = 4, n_2 = 12, λ_2 = 2, n = 2, m = 7.

With n = 3 and p = 2, incidence matrix of required singular group divisible design is obtained as following:

Table 1

0	0	1	0	1	1	1
0	1	0	1	0	1	1
0	1	1	1	1	0	0
1	0	0	1	1	0	1
1	0	1	1	0	1	0
1	1	0	0	1	1	0
1	1	1	0	0	0	1
0	0	1	0	1	1	1
0	1	0	1	0	1	1
0	1	1	1	1	0	0
1	0	0	1	1	0	1
1	0	1	1	0	1	0
1	1	0	0	1	1	0
1	1	1	0	0	0	1

Blocks of the Singular group divisible design are following

[4 5 6 7 11 12 13 14], [2 3 6 7 9 10 13 14]

[1 3 5 7 8 10 12 14], [2 3 4 5 9 10 11 12]

[1 3 4 6 8 10 11 13], [1 2 5 6 8 9 12 13]

[1 2 4 7 8 9 11 14].

Note that with different value of n and p, several Singular group divisible designs can be obtained.

3.4. Group Divisible Designs using Kronecker Product of Matrices

In this section we will develop another method of construction of partially balanced incomplete block design with two associate classes using Kronecker product of two BIB designs. This is discussed in theorem 3.4.

Theorem 3.4 Let D_1 is a balance incomplete block design with parameters v', b', r', k' and λ'. Let D_2 is another BIB design obtained using D_2= $v'E_{vv} + D_1$. Now the Kronecker product of two designs D_1 and D_2 given by D = D_1 x D_2 gives Group divisible designs with parameters v = $2v'$, b =b^2, r = $b'r'$, k = $2k'$, λ_1 = $b'\lambda'$, n_1= v'-1, λ_2= r^2 and n_2= v'.

Proof: Let us consider a balance incomplete block design with parameters v', b', r', k' and λ'. Let us call this design D_1. Next obtain another incomplete block design from D_2 = v' $E_{b'k'}$ +D_1, where dimension of both designs is same as well as the parameters; while varietal codes of both the designs D_1 and D_2 are different. Since in D_1 there is v' treatments and D_2 is obtained by adding v' treatments with D_1, so number of treatments for design D is $2v'$. Again, incomplete block design D is obtained using Kronecker product of the two designs D_1 and, D_2 which has b' blocks, so the number of blocks for the resulting incomplete block design are $b^{'2}$ and block sizes are obviously $2k'$. Similarly, r' blocks of D_1 is associated with b_1 block of D_2, hence the replication sizes of resulting design is $b'r'$, where as the secondary parameters are obvious. The resulting incomplete block design is Group divisible designs with parameters v = $2v'$, b =b^2, r = $b'r'$, k = $2k'$, λ_1 = $b'\lambda'$, n_1= v'-1, λ_2= r^2 and n_2= v'.

Example 3.4 Construct a semi-regular group divisible design with parameters v = 8, b = 36, r =18, k = 4, λ_1= 6, n_1 = 3, λ_2 = 9, and n_2 = 4.

Let us consider a balance incomplete block design with parameters $v' = 4$, b'=6 $r' = 3$, $k' = 2$ and $\lambda' = 1$. The blocks of this balanced incomplete block design D_1 are

1	1	1	2	2	3
2	3	4	3	4	4

Now, another incomplete block design D_2 is obtained from

$$D_2 = 4\begin{bmatrix}1 & 1\\1 & 1\\1 & 1\\1 & 1\\1 & 1\\1 & 1\end{bmatrix} + \begin{bmatrix}1 & 2\\1 & 3\\1 & 4\\2 & 3\\2 & 4\\3 & 4\end{bmatrix} = \begin{bmatrix}5 & 6\\5 & 7\\5 & 8\\6 & 7\\6 & 8\\7 & 8\end{bmatrix}$$

After taking Kronecker product of two designs matrices D_1 and D_2. We obtain an incomplete block design with 36 blocks which are:

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	3	3	3	3	3	3	4	4	4	4	4	4
5	5	5	6	6	7	5	5	5	6	6	7	5	5	5	6	6	7
6	7	8	7	8	8	6	7	8	7	8	8	6	7	8	7	8	8
2	2	2	2	2	2	2	2	2	2	2	2	3	3	3	3	3	3
3	3	3	3	3	3	4	4	4	4	4	4	4	4	4	4	4	4
5	5	5	6	6	7	5	5	5	6	6	7	5	5	5	6	6	7
6	7	8	7	8	8	6	7	8	7	8	8	6	7	8	7	8	8

This incomplete block design is a partially balance incomplete block design with two associate classes having parameters v = 8, b = 36, r = 18, k = 4, λ_1 = 6, n_1 = 3, λ_2 = 9, and n_2= 4. Again, we observed here that n = 4, m = 4 and the association matrices as $P_{ij}^1 = \begin{bmatrix}2 & 0\\0 & 4\end{bmatrix}$ and $P_{ij}^2 = \begin{bmatrix}0 & 3\\3 & 0\end{bmatrix}$.

Also, we verified that r – λ_1> 0 and rk – vλ_2 = 0 holds true so the resulting PBIB design is semi - regular group divisible design.

Moreover, we can further divide 36 blocks into 18 groups each of two blocks, where each group contain all the 8 treatments, therefore the resulting partially balanced incomplete block design is a resolvable semi- regular group divisible design. The 18 groups are following:

[(1 2 5 6), (3 4 7 8)]

[(1 2 5 7), (3 4 6 8)]

[(1 2 5 8), (3 4 6 7)]

[(1 2 6 7), (3 4 5 8)]

[(1 2 6 8), (3 4 5 7)]

[(1 2 7 8), (3 4 5 6)]

[(1 3 5 6), (2 4 7 8)]

[(1 3 5 7), (2 4 6 8)]

[(1 3 5 8), (2 4 6 7)]

[(1 3 6 7), (2 4 5 8)]

[(1 3 6 8), (2 4 5 7)]

[(1 3 5 8), (2 4 6 7)]

[(1 4 5 6), (2 3 7 8)]

[(1 4 5 7), (2 3 6 8)]

[(1 4 5 8), (2 3 6 7)]

[(1 4 6 7), (2 3 5 8)]

[(1 4 6 8), (2 3 5 7)]

[(1 4 7 8), (2 3 5 6)]

This is a 1- resolvable SRGD designs.

Remarks: Clatworthy (1973) has listed the table of two associate classes partially balanced incomplete block designs with the parameters r and k up to ten. Since r = 18, hence this PBIB design is not available in Claitworthy (1973) table and hence, we can claim that this SRGD design may be a new SRGD design.

Example 3.5 Semi -regular group divisible design with parameters v = 6, b = 9, r = 6, k = 4, λ_1 = 3, n_1= 2, λ_2 = 4 and n_2 = 3.

Consider a balanced incomplete block design with parameters v = b = 3, r = k = 2, and λ = 1. We can obtain semi -regular group divisible design with two- associate classes using the Kroneckker product of matrices. Here, the blocks of D_1 and D_2 designs are:

$$D_1 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad D_2 = 3\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 4 & 6 \\ 5 & 6 \end{bmatrix}$$

Now D = D_1 X D_2 gives the following blocks of an incomplete block design

[1 2 4 5], [1 2 4 6], [1 2 5 6],

[1 3 4 5], [1 3 4 6], [1 3 5 6],

[2 3 4 5], [2 3 4 6], [2 3 5 6]

This is a semi -regular group divisible design with parameters v = 6, b = 9, r = 6, k = 4, λ_1 = 3, n_1= 2, λ_2 = 4 and n_2 = 3, n = 3, m = 2. Further this design is rearranged in three groups, each group contain three blocks while each treatment occurs twice in the group. The groups are following

[(1 2 4 5), (1 3 5 6), (2 3 4 6)],

[(1 2 4 6), (1 3 4 5), (2 3 5 6)].

[(1 2 5 6), (1 3 4 6), (2 3 4 5)].

This is a 2–resolvable semi -regular group divisible design with parameters v = 6, b = 9, r = 6, k = 4, $\lambda_1 = 3$, n_1 = 2, λ_2 = 4 and n_2 = 3, n = 3, m = 2.

Example 3.6 Construct a group divisible design with parameters v = 8, b = 16, r =12, k = 6, λ_1 = 8, n_1 = 3, λ_2 = 9, and n_2 = 4.

Let us consider a balance incomplete block design with parameters $v' = 4$, $b = 4'$, $r' = 3$, k'=3 and $\lambda = 2'$. The blocks of this balanced incomplete block design D_1 are

1 2 3

1 2 4

1 3 4

2 3 4

Other incomplete block design D_2 is obtained from

$$D_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 3 & 4 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 7 \\ 5 & 6 & 8 \\ 5 & 7 & 8 \\ 6 & 7 & 8 \end{bmatrix}$$

Performing Kronecker product of two designs matrices D_1 and D_2 . We obtain an incomplete block design with 16 blocks which are following:

s

1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2
2	2	2	2	2	2	2	2	3	3	3	3	3	3	3	3
3	3	3	3	4	4	4	4	4	4	4	4	4	4	4	4
5	5	5	6	5	5	5	6	5	5	5	6	5	5	5	6
6_	6	7	7	6	6	7	7	6	6	7	7	6	6	7	7
7	8	8	8	7	8	8	8	7	8	8	8	7	8	8	

This incomplete block design is a partially balanced incomplete block design with two- associate classes having parameters v = 8, b = 16, r = 12, k = 6, λ_1 = 8, n_1 = 3, λ_2 = 9, and n_2 = 4. Again, we observed here that n = 4, m = 2 and the association matrices as $P_{ij}^{1} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$ and $P_{ij}^{2} = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$.

Also, we verified that r – λ_1> 0 and rk – vλ_2 = 0 holds true therefore, the resulting PBIB design is semi- regular group divisible design.

Remarks: Clatworthy (1973) has listed the table of two associate classes partially balanced incomplete block designs with the parameters r and k up to ten. Since r = 12, hence this PBIB design is not available in Claitworthy (1973) table and hence, we can claim that this SRGD design may be a new SRGD design.

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